Physics 618 2020

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Wigner's Theorem

Symmetry in QM — preserves probabilities of measurements

⇝ Map from pure states $\longrightarrow$ pure states preserving overlaps

Map: $\mathbb{P}\mathcal{H} := \left\{ \begin{array}{l} \text{rank 1} \\ \text{projectors} \\ P : \mathcal{H} \to \mathcal{H} \end{array} \right\}$

$\uparrow$

Hilbert space

$$\mathrm{Tr}\, P_1 P_2$$

$$f : \quad P \longrightarrow f(P)$$

$$\mathrm{Tr}\, f(P_1) f(P_2) = \mathrm{Tr}\, P_1 P_2$$

$\forall \; P_1, P_2 \in \mathbb{P}\mathcal{H}$

$\mathbb{P}\mathcal{H}$ is not linear

e.g. $\mathcal{H} = \mathbb{C}^2$ $\quad \mathbb{P}\mathcal{H} = \mathbb{CP}^1$

$$= S^2$$

$$\mathrm{Aut}(\mathcal{H}) = U(\mathcal{H}) \amalg AU(\mathcal{H})$$

$u \in U(\mathcal{H})$ $\quad \|u\psi\| = \|\psi\|$

$a \in AU(\mathcal{H})$ is $\mathbb{C}$-<u>antilinear</u>

$$a(\psi_1 + \psi_2) = a(\psi_1) + a(\psi_2)$$

but $z \in \mathbb{C}$ $\quad a(z\psi) = z^* a(\psi)$

a is anti<u>unitary</u> if in addition

$$\|a\psi\| = \|\psi\| \quad \forall \psi \in \mathcal{H}.$$

$\mathrm{Aut}(\mathcal{H})$ is a group.

$$\| a \circ u(\psi) \| = \| u(\psi) \| = \| \psi \|$$

$$a \circ u(z\psi) = z^* \, a \circ u(\psi)$$

$a \circ u$ and $u \circ a$ is antiunitary

but $a_1 \circ a_2$ is unitary because

$(z^*)^* = z.$

$$\phi : \mathrm{Aut}(\mathcal{H}) \longrightarrow \mathbb{Z}_2 = \{\pm 1\}$$

$$\phi(g) = \begin{cases} +1 & \text{if } g \text{ unitary} \\ -1 & \text{if } g \text{ antiunitary} \end{cases}$$

$\Rightarrow$ ϕ is a group hom.

$$1 \to U(\mathcal{H}) \to \mathrm{Aut}(\mathcal{H}) \xrightarrow{\phi} \mathbb{Z}_2 \to 1$$

$$\pi : \mathrm{Aut}(\mathcal{H}) \longrightarrow \mathrm{Aut}(QM)$$

$\mathrm{Aut}(QM)$ = group $f : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ preserving overlaps

$$\pi(u) : \quad P \longmapsto \underline{\underline{u P u^{\dagger}}} = \underline{\underline{u P u^{-1}}}$$

$$\pi(a) : \quad P \longmapsto \underline{\underline{a P a^{\dagger}}} = \underline{\underline{a P a^{-1}}}$$

If A is $\mathbb{C}$-antilinear $\quad P^2 = P$

$$(A^{\dagger}\psi_1, \psi_2) = (\psi_1, A\psi_2)^*$$

Compare A is $\mathbb{C}$-linear

$$(A^{\dagger}\psi_1, \psi_2) = (\psi_1, A\psi_2)$$

$$u^{\dagger} = u^{-1} \qquad a^{\dagger} = a^{-1}$$

By cyclicity of trace:

$$\mathrm{Tr}_{\mathcal{H}}\Big(\pi(u)(P_1)\ \pi(u)(P_2)\Big) = \mathrm{Tr}_{\mathcal{H}}(P_1 P_2)$$

$$\mathrm{Aut}(\mathcal{H}) \xrightarrow{\pi} \mathrm{Aut}(QM) \to 1$$

Wigner: This is <u>surjective</u>

i.e. any symmetry of a quant. system can be represented by a unitary or anti-unitary operator on $\mathcal{H}$.

$$\ker(\pi) = \{ z \cdot \mathbb{1} \mid |z| = 1 \} \cong U(1)$$

$$1 \to U(1) \to \mathrm{Aut}(\mathcal{H}) \xrightarrow{\pi} \mathrm{Aut}(QM) \to 1$$

In general we'll have dynamics

$H^{\dagger} = H$ Hamiltonian

central extensions

Some group G

$$
\begin{tikzcd}
 & U(1) \arrow[r] \arrow[d, equal] & \tilde{G} \arrow[r] \arrow[d] & G \arrow[d, "\rho"] & \ni g_1, g_2 \\
1 \arrow[r] & U(1) \arrow[r] & \mathrm{Aut}(\mathcal{H}) \arrow[r] & \mathrm{Aut}(QM) \arrow[r] & 1
\end{tikzcd}
$$

$\tilde{G}$ is a central extension of G

$$(*) \quad U(g_1)\,U(g_2) = \underbrace{c(g_1, g_2)}_{\text{phase}}\, U(g_1 g_2)$$

Linear operators are always associative.

$$\left(U(g_1)U(g_2)\right)U(g_3)$$
$$= U(g_1)\left(U(g_2)U(g_3)\right)$$

$$c(g_1,g_2)\,c(g_1g_2,g_3)$$
$$= c(g_1,g_2g_3)\,c(g_2,g_3)$$

Proj. Rep's are ubiquitous in QM
QFT ...

Quant. of bosons & fermions :

Heisenberg group ↑ Centrally extends translations on phase sp.

Clifford group ↑ Centrally extends $\mathbb{Z}_2^{2n}$

anomalies in QFT

Kac Moody groups Virasoro groups

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How To Classify Central Extensions

A - Abelian group

G - any group

$$1 \to A \xrightarrow{\iota} \tilde{G} \xrightarrow{\pi} G \longrightarrow 1$$

$$\iota(A) \subset Z(\tilde{G})$$

- If sequence splits then

$$\omega_g(a) = a \qquad a \in A \text{ central}$$

$$\tilde{G} \cong A \times G$$

- Choose a section

$$s : G \longrightarrow \tilde{G} \qquad \pi \circ s = Id_G$$

$$\pi\Big(\overbrace{s(g_1)\, s(g_2)\, s(g_1 g_2)^{-1}}^{\text{in } \ker(\pi) = \operatorname{im}(\iota)} \Big) = 1$$

$$s(g_1)\, s(g_2)\, s(g_1, g_2)^{-1} = \iota\big(f_s(g_1, g_2)\big)$$

$$(**)\quad \boxed{s(g_1)\, s(g_2) = \iota\big(\underbrace{f_s(g_1, g_2)}\big)\, s(g_1 g_2)}$$

$+$

for some function
$f_s : G\times G \to A$

$$(*)\quad s(g_1)\big(s(g_2)\, s(g_3)\big) = \big(s(g_1)\, s(g_2)\big)\, s(g_3)$$

$\Big\Downarrow$ $\forall\ g_1, g_2, g_3 \in G$

$$\boxed{\begin{aligned} &f_s(g_2, g_3)\, f_s(g_1, g_2 g_3) \\ &= f_s(g_1, g_2)\, f_s(g_1 g_2, g_3) \end{aligned}}$$

Cocycle identity.

Sple Consequence: $f(g,1) = f(1,g) = f(1,1)$
$f(g, g^{-1}) = f(g^{-1}, g)$

Def:

1. A 2-cochain on G w/ values in A

$$f : G \times G \longrightarrow A$$

That's it! $\{2\text{-cochains}\} = C^2(G,A)$

2. A 2-cocycle on G w/ values in A is $f \in C^2(G,A)$ s.t. f satisfies the cocycle identity.

$$\{2\text{-cocycles}\} := Z^2(G,A)$$

Rmk 1: Language from topology.

Rmk 2 $C^2(G,A)$ and $Z^2(G,A)$ are both Abllian groups

$$= (f_2 f_1)(g_1, g_2)$$
$$= f_2(g_1,g_2) f_1(g_1,g_2)$$

$$(f_1 \cdot f_2)(g_1,g_2) = \underbrace{f_1(g_1,g_2) \cdot f_2(g_1,g_2)}_{\text{prod. in } A}$$

Choosing a section $\hat{s}$

$$
\begin{tikzcd}
1 \arrow[r] & A \arrow[r] & \tilde{G} \arrow[r, "\pi"'] & G \arrow[r] \arrow[l, dashed, bend right, "s"'] \arrow[l, dashed, bend right=60, "\hat{s}"'] & 1
\end{tikzcd}
$$

defines $f_s \in Z^2(G, A)$

Choose a different section $\hat{s}$?

how is $f_{\hat{s}}$ related to f_s?

$$\pi(\hat{s}(g)) = \pi(s(g)) \Longrightarrow$$

$$\hat{s}(g) = z(t(g))\, s(g)$$

for some function $t: G \longrightarrow A$.

$$\boxed{f_{\hat{s}}(g_1, g_2) = f_s(g_1, g_2)\, \underbrace{t(g_1)\, t(g_2)\, t(g_1 g_2)^{-1}}_{\text{A Abelian} \Rightarrow \text{order doesn't matter}}}$$

Used here that it is a central extension

Def: We say two cocycles f and $\hat{f}$ differ by a <u>coboundary</u> if $\exists$ $t: G \longrightarrow A$ s.t.

$$\hat{f}(g_1, g_2) = f(g_1, g_2) \frac{t(g_1) t(g_2)}{t(g_1 g_2)}$$

Equiv. Rel: $\hat{f} \sim f$ if they differ by a cocycle. $[f]$ = equiv.

Def: $H^2(G, A)$ = group cohomology set of equiv. classes of 2-cocycles.

Theorem: Isomorphism classes of <u>central</u> extensions of G by A are in 1-1 correspondence with elements of $H^2(G, A)$.

$$1 \to A \xrightarrow{\iota} \widetilde{G} \xrightarrow{\pi} G \to 1 \in \mathcal{E}xt$$

Choose a section $\Longrightarrow$ $[f_s]$ indpt of section.

$$\mathcal{E}xt \longrightarrow H^2(G,A)$$

$$\boxed{\mathcal{E}xt / \text{isom.} \longrightarrow H^2(G,A)}$$

$$\begin{tikzcd}
 & & \widetilde{G} \arrow[dd, "\psi"] \arrow[dr, "\pi"] & & \\
1 \arrow[r] & A \arrow[ur, "\iota"] \arrow[dr, "\iota'"'] & & G \arrow[r] \arrow[ul, dashed, bend right, "s"'] & 1 \\
 & & \widetilde{G}' \arrow[ur, "\pi'"'] & &
\end{tikzcd}$$

$\psi : \widetilde{G} \longrightarrow \widetilde{G}'$ isom.

$s' = \psi \circ s$ is a section of π'

$$s'(g_1)\, s'(g_2) = \iota'(f_s(g_1, g_2))\, s'(g_1 g_2)$$

$\Rightarrow$ Same cocycle $f_{s'} = f_s$.

Suppose $[f] \in H^2(G, A)$

$\tilde{G} = A \times G$ as a set

with group structure

$$(a_1, g_1) \cdot (a_2, g_2)$$

$$:= (a_1 a_2 f(g_1, g_2), g_1 g_2)$$

you check: This is a valid group law. Associative $\Leftrightarrow$ f is a cocycle.

$$[f] = [\hat{f}] \leftarrow$$

then $\exists$ isomorphism of extensions

$$1 \to A \xrightarrow{\imath} \hat{G} \xrightarrow{\pi} G \to 1$$

$\pi : (a, g) \to g$

$\imath : a \longrightarrow (a, 1)$

$$\mathrm{Ext}/\mathrm{isom} \rightleftarrows H^2(G,A)$$

These are inverses. $\blacksquare$

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① Central extensions & projective reps.

$$\rho(g_1)\rho(g_2) = \underbrace{c(g_1,g_2)}_{\text{satisfies cocycle relation}}\rho(g_1 g_2)$$

$c(g_1,g_2)$ are valued in $U(1)$

$\therefore$ define

$$1 \to \underline{U(1)} \to \tilde{G} \to \underline{G} \to 1 \qquad \tilde{G} \cong U(1)\times G$$

using this cocycle.

$$\tilde{\rho}(z,g) = z\,\rho(g)$$

is a TRUE REPRESENTATION of $\tilde{G}$.

you think you have a symmetry group G. Start to define operators $U(g)$ on $\mathcal{H}$ discover

$$U(g_1)U(g_2) = \underbrace{c(g_1, g_2)}_{\text{can't be removed by phase redefinition of } U(g)} U(g_1 g_2)$$

True symmetry group is $\widetilde{G}$, the central extension.

<u>Exple</u>: Rotation group $SO(3)$

$1/2$- spin rep's projective.
True symmetry group is $SU(2)$.

$$U(1) \to \widetilde{G} \to SO(3) \to 1$$
$$\cup$$
$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1$$
$$c \in \pm 1$$

② $H^2(G,A)$ this is in fact a group.

$$[f_1]\cdot[f_2] = [f_1 \cdot f_2]$$

Check this is well-defined //

$$[\hat{f}_1]\cdot[\hat{f}_2] = [\hat{f}_1\cdot\hat{f}_2]$$

$$1 \to A \xrightarrow{i_1} \tilde{G}_1 \xrightarrow{\pi_1} G \to 1$$

$$1 \to A \xrightarrow{i_2} \tilde{G}_2 \xrightarrow{\pi_2} G \to 1$$

$$\begin{tikzcd}
1 \arrow[r] & A\times A \arrow[r] & \tilde{G}_1 \times \tilde{G}_2 \arrow[r] & G\times G \arrow[r] & 1 \\
& A\times A \arrow[r] & \text{pullback} \arrow[r] & G \arrow[u, "\Delta"] &
\end{tikzcd}$$

(3) Trivial vs. Trivializable

Trivial cocycle $f(g_1, g_2) = 1$

Trivializable cocycle $[f] = [1]$

$$f(g_1, g_2) = \frac{t(g_1)\, t(g_2)}{t(g_1 g_2)} \quad \text{for some } t$$

t is called the "trivialization of f"

t_1, t_2 are two trivializations

$$t_1(g) = t_2(g)\, \phi(g) \qquad \phi : G \to A$$

is a group homomorphism

④ Analogy to Gauge Theory

$$A'_\mu(x) = A_\mu(x) - i g^{-1}(x) \partial_\mu g(x)$$

g : spacetime $\longrightarrow U(1)$

$\hat{f}$ vs. f change of gauge.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$F = \frac{1}{2!} F_{\mu\nu} dx^\mu \wedge dx^\nu$
Curvature of $\nabla = d + A$

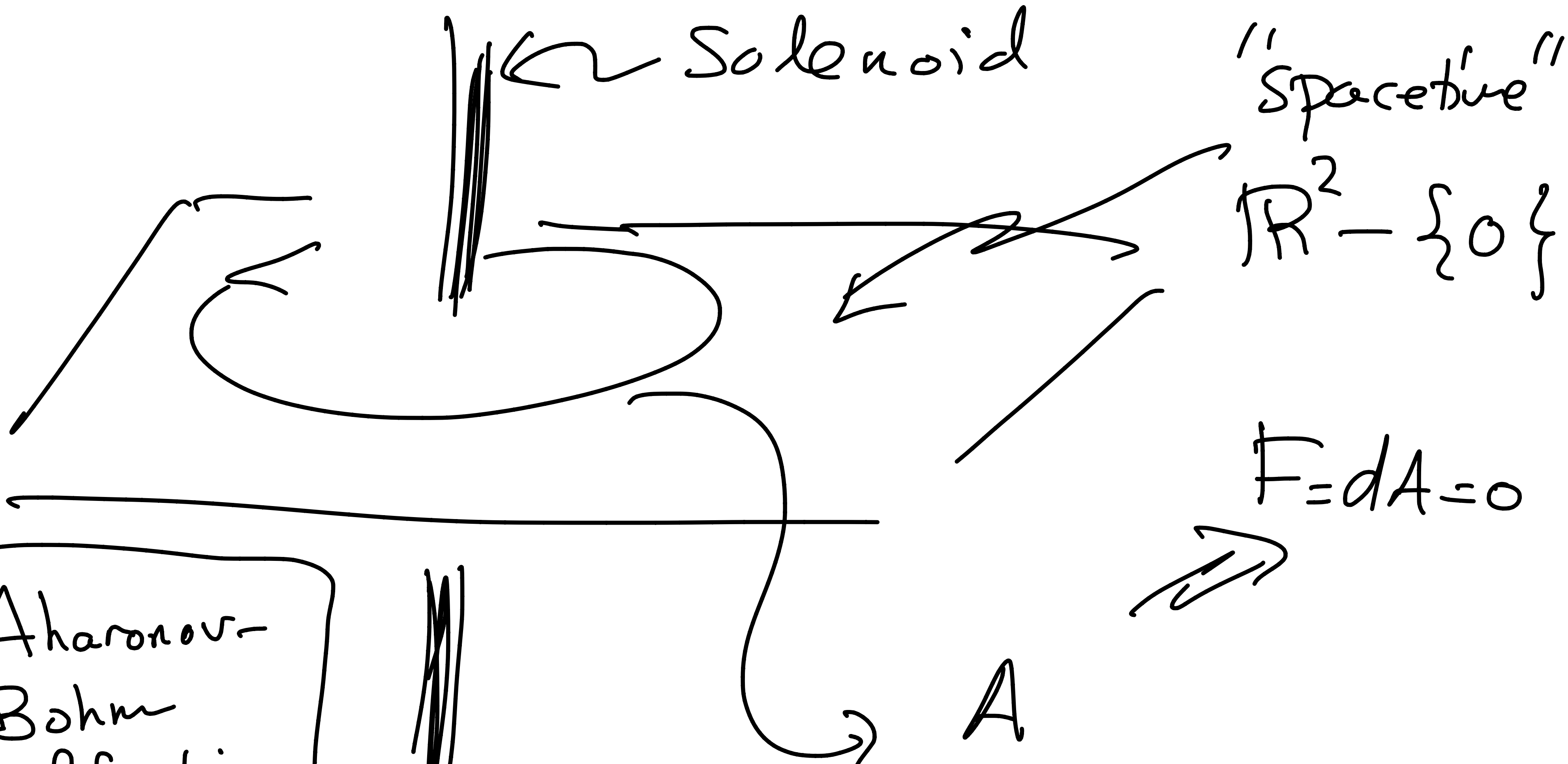
Ask: $F_{\mu\nu} = 0 \Rightarrow A_\mu$ is gauge equiv. to zero.?

If $\gamma \subset$ Spacetime is a closed loop

$$\oint_\gamma A := \int_0^1 A_\mu(x(t)) \frac{dx^\mu(t)}{dt} dt$$

$$\boxed{\mathrm{Hol}(\gamma, A) = \exp\left(i \oint_\gamma A\right)}$$ is also gauge invariant

It can happen that $F = 0$ but $\mathrm{Hol}(\gamma, A)$ is not $= 1$



$F = dA = 0$

Aharonov-Bohm effect

$$\oint A = 2\pi B$$

$$\exp(2\pi i B) \quad B \notin \mathbb{Z}$$

<u>Theorem:</u> Gauge equiv. class of A is determined by the function

$$\mathrm{Hol} : \text{closed loops} \longrightarrow U(1)$$

(5) Often useful to choose gauges to simplify a problem.

Let's use coboundaries to simplify a cocycle. $f \in Z^2(G,A) \Rightarrow$

$$f(1,g) = f(g,1) = f(1,1)$$

Change by a coboundary t

$$f^{(1)}(1,1) = f(1,1)\frac{t(1)t(1)}{t(1\cdot 1)}$$

$$= f(1,1)t(1)$$

So if we choose $t(1) = f(1,1)^{-1}$ then we set

$$f^{(1)}(1,1) = f^{(1)}(1,g) = f^{(1)}(g,1) = 1$$

"normalized cocycle"

further simplify, so long as $t^{(1)}(1) = 1$. partial choice of gauge

$$f^{(2)}(g,g^{-1}) = f^{(1)}(g,g^{-1}) \frac{t^{(1)}(g)\,t^{(1)}(g^{-1})}{t^{(1)}(g\cdot g^{-1})}$$

$$= f^{(1)}(g,g^{-1})\, t^{(1)}(g)\, t^{(1)}(g^{-1})$$

$$G = \underset{\{g \mid g^2=1\}}{\underset{\shortparallel}{\mathrm{Inv.}}} \amalg \underset{\{g \mid g^2\neq 1\}}{\underset{\shortparallel}{\mathrm{NonInv.}}}$$

$$\mathrm{NonInv} = S_1 \amalg S_2$$

no two elements of S_1 are related by $g \to g^{-1}$

<u>Can</u> Choose $t^{(1)}(g)$ $g \in S_1$ so that

$$f^{(2)}(g,g^{-1}) = 1 \quad \forall\, g \in \mathrm{NonInv.}$$

If $g \in \mathrm{Inv}$

$$\underset{\substack{\| \\ g'}}{f^{(2)}(g, g)} = f^{(1)}(g, g)\frac{\tilde{t}(g)^2}{\tilde{t}(g^2)}$$

$$= f^{(1)}(g,g)\,\tilde{t}(g)^2$$

We haven't fixed $\tilde{t}$ on Inv yet.

If $f^{(1)}(g,g)$ is NOT a perfect square in the group we cannot gauge it to $\mathbb{1}$.

Example of a gauge invariant obstruction to putting $f \to 1$.

If $f(g,g)$ is not a perfect square for some involution $g \in G$ then f is a nontrivial cocycle.

Example 1: Central Extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_2$

$$1 \to \mathbb{Z}_2 \xrightarrow{\imath} \tilde{G} \xrightarrow{\pi} \mathbb{Z}_2 \to 1$$

$$\mathbb{Z}_2 = \{1, \sigma_1\}, \quad \sigma_1^2 = 1 \qquad\qquad \mathbb{Z}_2 = \{1, \sigma_2\}, \quad \sigma_2^2 = 1$$

WLOG $f(1,1) = f(1,\sigma_2) = f(\sigma_2, 1) = 1$

$f(\sigma_2, \sigma_2)$? $\qquad \sigma_2^2 = 1$

$f(\sigma_2, \sigma_2) = 1$ Trivial cocycle

$$1 \to \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \to 1$$

$f(\sigma_2, \sigma_2) = \sigma_1$ $\qquad$ σ_1 is not a perfect square!

$$\tilde{G} \cong \mathbb{Z}_4$$

$$(1,\sigma_2)\cdot(1,\sigma_2) = (f(\sigma_2,\sigma_2), \sigma_2^2=1)$$

$$= (f(\sigma_2,\sigma_2), 1)$$

$$= (\sigma_1, 1) \quad \text{order } 2$$

So $(1,\sigma_2)$ has order 4 and generates the whole group.

---x---

<u>Example 2:</u> Ext's of $\mathbb{Z}_p$ by $\mathbb{Z}_p$

p = prime.

$$1 \longrightarrow \mathbb{Z}_p \longrightarrow G \longrightarrow \mathbb{Z}_p \longrightarrow 1$$

Methods of topology show that

$$H^2(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$$

But $\rightsquigarrow$ puzzle:

Sylow + class eq. $\Rightarrow$ there are only two isomorphim types of groups of order p^2 !!

$$1 \to \mathbb{Z}_p \to \tilde{G} \to \mathbb{Z}_p \to 1$$

Then $\tilde{G} \cong \mathbb{Z}_p \times \mathbb{Z}_p$

or $\tilde{G} \cong \mathbb{Z}_{p^2}$

only two possibilities

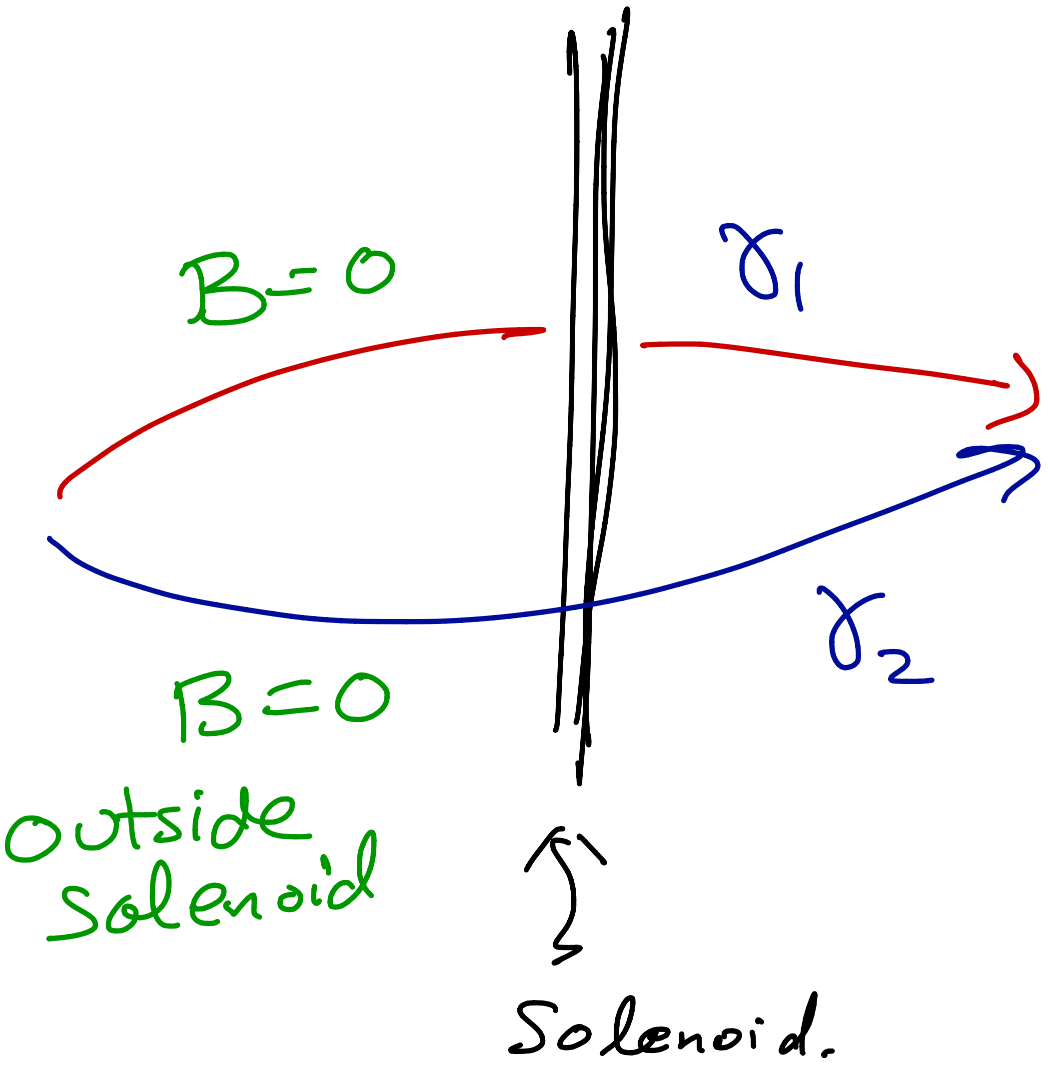
How can there be p different isomorphism classes of central extensions for p an odd prime?

Answer: Friday

Aharanor-Bohm effect.

In q.m. the phase picked up by a charged particle traveling in an electromagnetic field is

$$\exp i \int_\gamma eA \qquad e = \text{electric charge.}$$



Reps of $U(1)$
$e \in \mathbb{Z}$

$$\psi_1 = \exp\left(i \oint e A\right) \psi_2$$

$\Longrightarrow$ nontrivial interference

$\psi_1 + \psi_2$ $\quad$ <u>AB effect</u>.

Flat gauge fields (flat: $F_{\mu\nu} = 0$) can nevertheless have nontrivial holonomy.